

# An Analytical Study in Coupled Map Lattices of Synchronized States and Travelling Waves, and of their Period-Doubling Cascades

M<sup>a</sup> Dolores Sotelo Herrera<sup>a</sup> & Jesús San Martín<sup>a,b</sup>

<sup>a</sup> Departamento de Matemática Aplicada, E.U.I.T.I., Universidad  
Politécnica de Madrid. Ronda de Valencia 3, 28012 Madrid Spain

<sup>b</sup> Departamento de Física Matemática y de Fluidos, U.N.E.D. Senda del  
Rey 9, 28040 Madrid Spain

Corresponding author: jsmdf@uned.es

## Abstract

Several theorems are demonstrated that determine the sufficient conditions for the existence of synchronized states (periodical and chaotic) and also of travelling waves in a CML. Also are analytically proven the existence of period-doubling cascades for the mentioned patterns. The temporal state of any oscillators are completely characterized. The given results are valid for a number of arbitrary oscillators whose individual dynamics is ruled by an arbitrary  $C^2$  function.

Systems showing patterns as a consequence of the interaction among their diverse components are really frequent, in any field that one can imagine: neuronal activity within the brain, or the function of organs as a whole within the body, drivers on a motorway, birds flying in a group, a network of computers, coupled lasers, crystal growth, etc.

The result of the interaction of the individual elements generates structures that manifest in the system as a whole. In these processes, one should consider two things: the behavior of any individual and the interaction among them. If we consider the traffic example, it is clear that the behavior of an individual driver, that is his decision to drive in a particular way or another, is certainly different when there are few cars on a motorway or when there is a traffic jam (in which case he will be guided by traffic patterns).

Broadly speaking, all of these systems consist of a group of elements coupled by some kind of process, and at the same time, every element of the group is ruled by its own local dynamics. The understanding of such systems is extraordinarily complicated, since there are no particular mathematical tools developed to study them. One way to confront this problem is to discretize spatial and temporal variables as well as to fix inter-individual interactions as well as the individual dynamics. The result is a Coupled Map Lattice (CML) [1]: a chain of coupled elements (called oscillators), each situated on a discrete point of the lattice, whose individual dynamics is ruled by a discrete map. Despite the spatial and temporal variables are discretized, state variables remain continuous.

In the last few years, CML have been extensively studied since the work of Kaneko and collaborators [2, 3, 4, 5, 6], and from the beginning, they have

shown themselves to be exceptional modelling spatially extended systems. The use of this study has been extended into diverse scientific branches with an extraordinary variety of applications in physics, biology, chemistry, social sciences, and engineering modeling. [7, 8]

A typical evolution equation for a CML [1] is given by

$$X_i(n+1) = (1-\alpha)f(X_i(n)) + \frac{\alpha}{m} \sum_{j=1}^m f(X_j(n)) \quad (1)$$

$$i = 1, \dots, m$$

where  $X_i(n)$  represents the state of the oscillator located at node “i” of the lattice, in the instant “n”. The parameter  $\alpha$  weights the coupling among oscillators. Periodic conditions are assumed in the boundaries, given as

$$X_i(n) = X_{i+m}(n) \quad \forall i$$

Depending on the value of  $\alpha$ , the system behavior changes from the independent evolution of each oscillator (for  $\alpha = 0$ ) up to a mean field approach (for  $\alpha = 1$ ). For intermediate values  $0 < \alpha < 1$  the system is ruled by both local and global mechanisms.

The general form of the coupling term is given by

$$\frac{\alpha}{m} \sum_{j=1}^m w_{ij} f(X_j(n))$$

where the  $w_{ij}$  measure the weights between the  $j$ -th oscillator and the  $i$ -th one. To achieve a symmetrical and spatially invariant coupling, it is usually taken  $w_{ij} = \bar{w}_{|i-j|}$ . Sometimes, the coupling term will be written as

$$\frac{\alpha}{m} \sum_{j=1}^m f(X_j(n))$$

(mean field), or

$$\frac{1}{2} [f(X_{j-1}(n)) + f(X_{j+1}(n))]$$

(nearest-neighbor coupling). However, this last description is not adequate when we are dealing with a supercritical bifurcation threshold, because the coherence lengths are usually quite large [9]. Given that, in this paper, we want to study bifurcations in CML, we will use the mean field approach.

Another important point, that must be considered, is the updating of oscillators; they can be synchronous (all oscillators are updated simultaneously) or asynchronous (oscillators are updated one at a time) [10, 11]. Choosing one or the other depends on whether oscillators communicate among them much quicker than the updating time of the system as a whole, which is ruled by the evolution equation (1). In this paper we will refer to synchronous systems.

In the scientific literature, the majority of the results, referring to CML, are numerical results, as we will see later. The awesome richness of numerical results is restricted by a fixed and finite set of parameter values, and a finite number of oscillators in the CML, which supposes a limitation for adequate understanding of certain phenomena. In particular, the transition

to chaos by period duplication needs the period to tend to infinity. It is also necessary the number of oscillators to be infinite, in a finite region, for the understanding of the onset of turbulence in fluids and plasmas; otherwise, there would be a cutoff in the wave numbers that could be studied because the lattice would have a finite spatial resolution. Mathematical proofs would be desirable to characterize synchronized states, traveller wave bifurcations and other behaviours. Fortunately, numerical results point out us what to look for and where.

In this paper, analytical proofs, in CML, of the existence of synchronized states and travelling waves will be given. It will be proved that both patterns will go under a period doubling cascade as  $f$ , in (1), does. These behaviours will be completely characterized, giving analytical expressions of the temporal evolution of every oscillator.

The fixed points of CML, generated in period-doubling cascades, will be essentially the fixed points of  $f^{m2^k}$  ( $m$  number of oscillators in CML). As  $f$  determines the individual dynamics (see (1)), what is shown is the emergence of global properties from the local dynamics of a single oscillator.

We have tried to keep the widest generality in the results; therefore, theorems have been proved using an arbitrary  $C^2$  function  $f(x; r)$ , instead of working with the logistic equation (or any topologically conjugated functions) as usual.

Perturbative methods will be used to obtain analytical solutions. The inversion of functional matrices of arbitrary size is fundamental in the proofs of the theorems; given that whenever the inverse matrix exists, it is unique, it will not be necessary to explain the calculation leading to it: it will be

enough to check that the proposed matrix (in the corresponding theorem) is the inverse matrix one was looking for. The matrices appearing during the demonstration process will not be circulant; therefore, usual analytical inversion processes of circulant matrix inversion will not be valid.

This paper is organized as follows. First, synchronized states will be considered, this solution being quite straightforward, it will indicate how to face up to the more complicated travelling waves in the next section. Both results will be used to study the period-doubling cascades of the patterns. The paper concludes with a section indicating connections of this work with other researchs.

## 1 Regular and chaotic synchronization

In this section straightforward analytical results will be presented for synchronization in CML, that is, for all the oscillators having the same value at anytime. This is a striking behaviour, in particular when chaotic synchronization is produced, where chaotic systems are very sensitive to perturbations and it is supposed that any slight modification generated by the coupling of the oscillators of CML would destroy the synchronization. The mathematical approach to this problem is far from being unique [12].

Let

$$X_i(n+1) = (1-\alpha)f(X_i(n)) + \frac{\alpha}{m} \sum_{i=1}^m f(X_i(n)) \quad i = 1, \dots, m \quad (2)$$

be the CML, with  $m$  oscillators, being  $\alpha$  the coupling parameter and  $f(x)$  a function depending on a parameter  $r$ , in function of which the system  $y_{n+1} = f(y_n; r)$  shows fixed points for some arbitrary period  $p$ .

## 1.1 Fixed points of the system. Stationary synchronized state

It is straightforward to get the fixed points of the system. If the function  $f(x)$  has a fixed point in  $x^*$  then  $(x^*, x^*, \dots^m, x^*)$  will be a fixed point of the system given by (2), since if

$$X_i(n) = x^* \quad i = 1, \dots, m$$

it turns out that

$$f(X_i(n)) = f(x^*) = x^* = X_i(n) \quad i = 1, \dots, m$$

and

$$X_i(n+1) = (1 - \alpha)f(x^*) + \frac{\alpha}{m} \sum_{i=1}^m f(x^*) = f(x^*) = x^*$$

$$i = 1, \dots, m$$

as it was wanted to prove.

It is then deduced that:

$$X(n) = (x^*, x^*, \dots^m, x^*) \tag{3}$$

is a stationary synchronized state of the system.

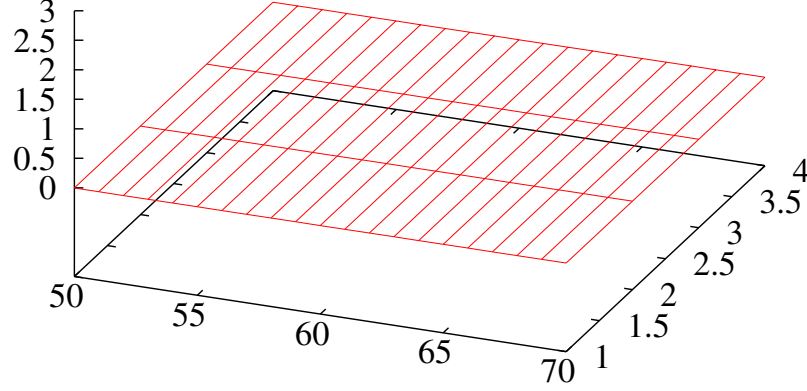


Figure 1: Stationary synchronized state. CML with  $f(x; r) = rx(1 - x)$ ,  $r = 1.0$ ,  $\alpha = 0.1$  and  $\varepsilon = 0.1$

It is observed that if chosen  $x^*$  and  $r$  for  $f(x^*; r)$  to determine a periodical or chaotic evolution of  $x^*$ , then the result would be that CML would have correspondingly periodical or chaotic synchronization, being this a proof of the existence of synchronized states both periodical and chaotic. In contrast from the CML with nearest neighbour coupling, there is not an upper limit in the number of oscillators “ $m$ ” such that stable synchronous chaotic state exists [13]. See figures 1-4.

Let us now study the linear stability of fixed points, where the eigenvalues of jacobian matrix will be calculated.

The jacobian matrix is given by:



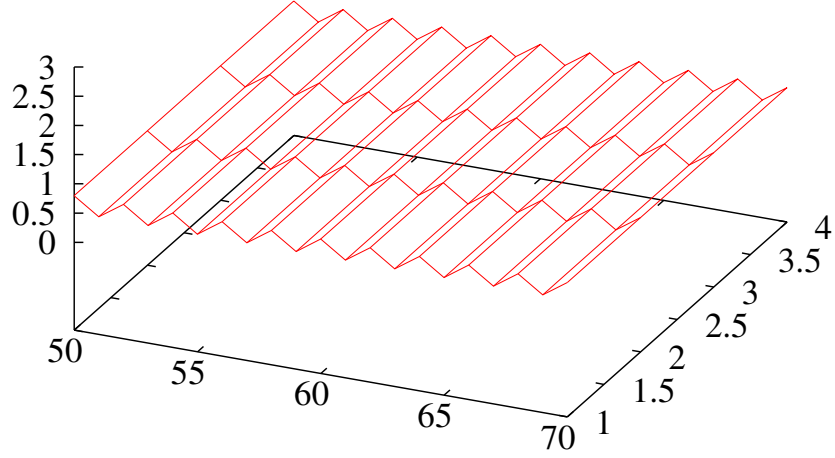


Figure 2: Period-2 synchronized state. CML with  $f(x; r) = rx(1 - x)$ ,  $r = 3.2$ ,  $\alpha = 0.1$  and  $\varepsilon = 0.1$

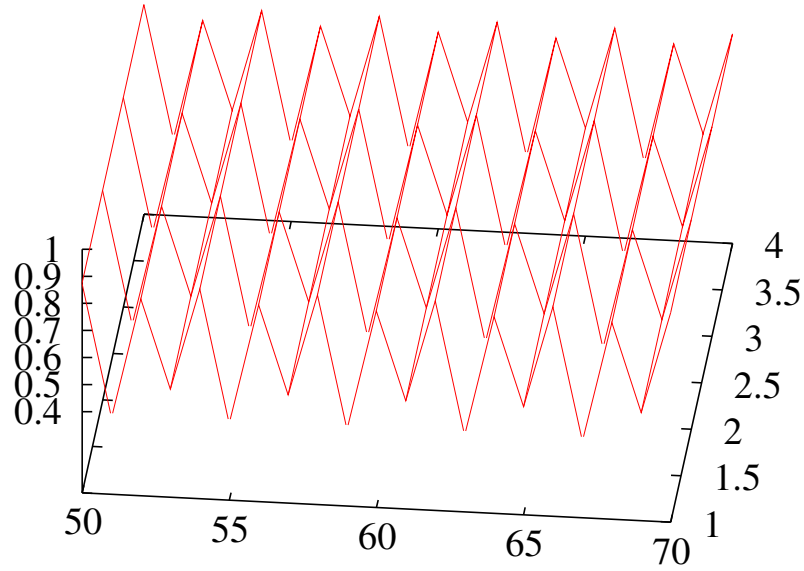


Figure 3: Period-4 synchronized state. CML with  $f(x; r) = rx(1 - x)$ ,  $r = 3.4985$ ,  $\alpha = 0.1$  and  $\varepsilon = 0.1$

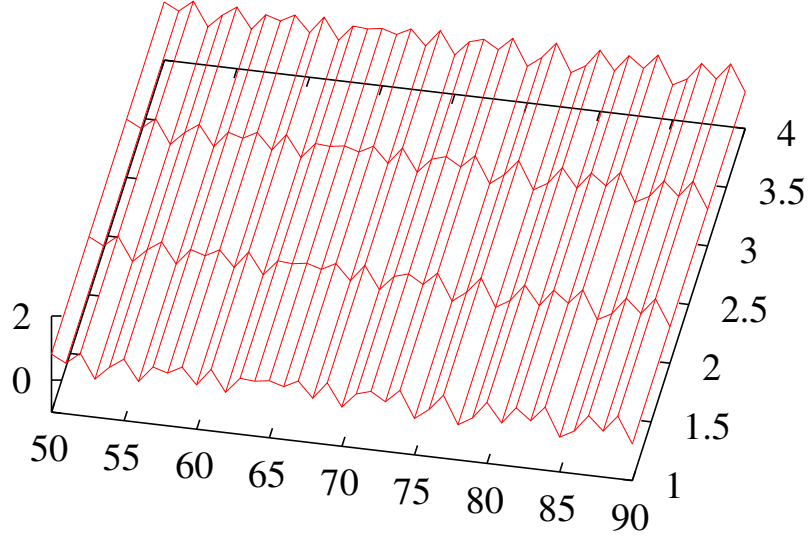


Figure 4: Chaotic synchronized state. CML with  $f(x; r) = rx(1-x)$ ,  $r = 3.9$ ,  $\alpha = 0.1$  and  $\varepsilon = 0.1$

$$\left( \frac{\partial X_i(n+1)}{\partial X_j(n)} \right)_{X^*} = \begin{pmatrix} (1 - \frac{m-1}{m}\alpha) & \frac{\alpha}{m} & \cdots & \frac{\alpha}{m} \\ \frac{\alpha}{m} & (1 - \frac{m-1}{m}\alpha) & \cdots & \frac{\alpha}{m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\alpha}{m} & \frac{\alpha}{m} & \cdots & (1 - \frac{m-1}{m}\alpha) \end{pmatrix} f'(x^*)$$

where  $X^* = (x^*, x^*, \dots, x^*)$  and its eigenvalues are

$$\begin{cases} \lambda = f'(x^*) & \text{single} \\ \lambda = (1 - \alpha)f'(x^*) & \text{multiplicity } (m - 1) \end{cases}$$

therefore, the fixed point given by (3), or what would be the same, the stationary schronized state, would be stable whenever  $x^*$  is a stable fixed point of  $f(x)$ .

## 1.2 Periodical synchronized states

The existence of periodical synchronized states is reflected in theorem 1, shown below. The way to proceed with this proof is similar to the one used to obtain the stationary synchronized state.

**Theorem 1.** Let  $\{x_1^*, x_2^*, \dots, x_p^*\}$  be a  $p$ -periodic orbit of  $C^1$  function  $f$ . Then the CML given by

$$X_i(n+1) = (1-\alpha)f(X_i(n)) + \frac{\alpha}{m} \sum_{i=1}^m f(X_i(n)) \quad i = 1, \dots, m$$

i) shows a synchronized state of the same period as the function  $f$ .

The synchronized states are as follows

$$(x_j^*, x_j^*, \dots^m, x_j^*)_{j=1, \dots, p}$$

ii) the synchronized states  $(x_j^*, x_j^*, \dots^m, x_j^*)_{j=1, \dots, p}$  have the same stability as the fixed point  $x_j^*$  of  $f^p$ ,  $j = 1, \dots, p$ .

### Proof

i) Taking, for a given time  $n$ ,

$$X_i(n) = x_1^* \quad i = 1, \dots, m$$

results in

$$f^p(X_i(n)) = X_i(n) \quad i = 1, \dots, m$$

with the first iteration of CML being:

$$X_i(n+1) = (1-\alpha)f(x_1^*) + \frac{\alpha}{m} \sum_{j=1}^m f(x_1^*) = f(x_1^*) \quad i = 1, \dots, m$$

and the  $p$ -th iteration being:

$$\begin{aligned} X_i(n+p) &= (1-\alpha)f(X_i(n+p-1)) + \frac{\alpha}{m} \sum_{j=1}^m f(X_j(n+p-1)) \\ &= f^p(x_1^*) = x_1^* \quad i = 1, \dots, m \end{aligned}$$

As a result, the CML shows  $p$  fixed points  $(x_j^*, x_j^*, \dots, x_j^*)_{j=1, \dots, p}$  of period  $p$ , which constitute one synchronized state of the same period. These orbits of period  $p$  constitute patterns of the CML.

- ii) To study the stability of the fixed points  $(x_j^*, x_j^*, \dots, x_j^*)_{j=1, \dots, p}$  it is enough to perform it in  $X_1^* = (x_1^*, x_1^*, \dots, x_1^*)$ , because  $f^{p'}(x_j^*)$  has the same value for every fixed point  $x_j^*$  of the  $p$ -periodic orbit.

Let us calculate the eigenvalues of the jacobian matrix of the  $p$ -th iterate in that point.

To calculate this Jacobian matrix, one must observe the following,

applying the chain rule:

$$\left( \frac{\partial X_i(n+p)}{\partial X_j(n)} \right)_{X_1^*} = \begin{pmatrix} (1 - \frac{m-1}{m}\alpha) & \frac{\alpha}{m} & \cdots & \frac{\alpha}{m} \\ \frac{\alpha}{m} & (1 - \frac{m-1}{m}\alpha) & \cdots & \frac{\alpha}{m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\alpha}{m} & \frac{\alpha}{m} & \cdots & (1 - \frac{m-1}{m}\alpha) \end{pmatrix} f'(x_p^*) \left( \frac{\partial X_i(n+p-1)}{\partial X_j(n)} \right)_{X_1^*}$$

finally as a result:

$$\left( \frac{\partial X_i(n+p)}{\partial X_j(n)} \right)_{X_1^*} = \begin{pmatrix} \frac{1}{m} + \frac{m-1}{m}(1-\alpha)^p & \frac{1}{m} - \frac{1}{m}(1-\alpha)^p & \cdots & \frac{1}{m} - \frac{1}{m}(1-\alpha)^p \\ \frac{1}{m} - \frac{1}{m}(1-\alpha)^p & \frac{1}{m} + \frac{m-1}{m}(1-\alpha)^p & \cdots & \frac{1}{m} - \frac{1}{m}(1-\alpha)^p \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m} - \frac{1}{m}(1-\alpha)^p & \frac{1}{m} - \frac{1}{m}(1-\alpha)^p & \cdots & \frac{1}{m} + \frac{m-1}{m}(1-\alpha)^p \end{pmatrix} \prod_{i=1}^p f'(x_i^*)$$

with eigenvalues

$$\left\{ \begin{array}{ll} \lambda = \prod_{i=1}^p f'(x_i^*) = f^{p'}(x_1^*) & \text{single} \\ \lambda = (1-\alpha)^p \prod_{i=1}^p f'(x_i^*) = (1-\alpha)^p f^{p'}(x_1^*) & \text{multiplicity } (m-1) \end{array} \right. \quad (4)$$

So,  $(x_1^*, \dots, x_1^*)$  is a stable point of the CML, and therefore, it is

in a stable synchronized state of period  $p$ , whenever  $x_1^*$  is the stable fixed point  $f^p$ . Furthermore, as  $f^{p'}(x_1^*) = f^{p'}(x_j^*) \quad j = 1, \dots, m$  all points have the same stability.

Keep in mind that if in **Theorem 1**  $p = 1$ , then the stationary synchronized state previously studied is recovered; and because of this, it will undergo the period-doubling process that will be described in what follows.

### 1.3 Period doubling cascade of periodic synchronized states

One would expect that if the function  $f^p$ , from **Theorem 1**, undergoes a period doubling cascade, then the CML given by (2) shows a duplication cascade in the synchronized states of period  $p$  derived in **Theorem 1**.

**Theorem 2.** The synchronized states of period  $p$  given by Theorem 1 undergo a period-doubling cascade as does  $f^p$ .

#### Proof

The proof is straightforward using Theorem 1, simply using successive substitution of  $p$  by  $p \cdot 2, p \cdot 2^2, \dots, p \cdot 2^n, \dots$  every time that the  $p$ -periodic orbit of  $f$  undergoes a period doubling bifurcation according to that theorem.

**Note:** See figures 5 and 6.

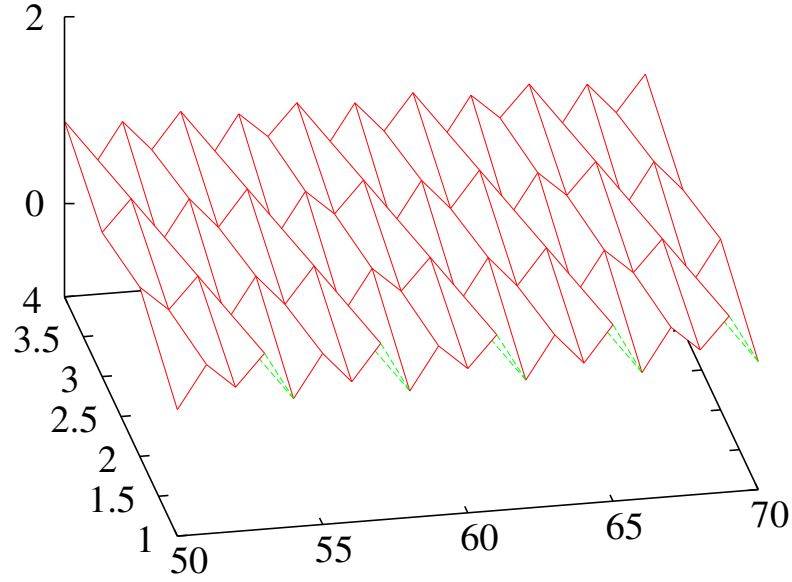


Figure 5: Period-4 travelling wave. CML with  $f(x; r) = rx(1 - x)$ ,  $r = 3.55464$ ,  $\alpha = 0.1$  and  $\varepsilon = 0.001$

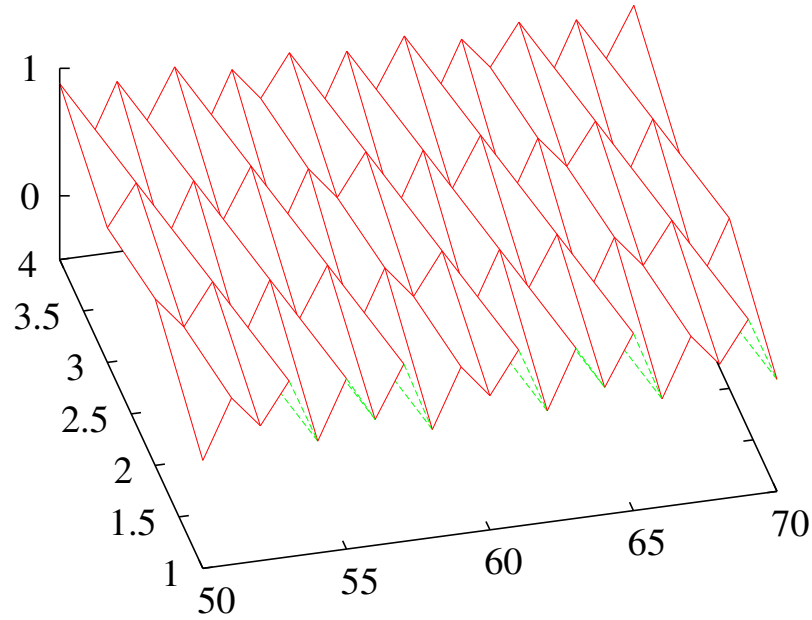


Figure 6: Period-8 travelling wave. CML with  $f(x; r) = rx(1 - x)$ ,  $r = 3.566667$ ,  $\alpha = 0.1$  and  $\varepsilon = 0.001$

## 1.4 A nonexistence theorem

It has been proven, in Theorem 1, the existences of  $p$ -period synchronized states in the CML, formed by the points of  $p$ -periodic orbit of  $f$ .

One may ask whether the  $n$ -tuple of the form

$$\begin{aligned}(X_1(n), X_2(n), \dots, X_p(n)) &= (x_j^*, f(x_j^*), \dots, f^{p-1}(x_j^*)) \\ f^p(x_j^*) &= x_j^* \quad j = 1, \dots, p\end{aligned}$$

that indicates that every oscillator is positioned in the successive points of the  $p$ -periodic orbit, generates a pattern of period  $p$  in the CML; that is to say, a travelling wave. Nevertheless, this presumption is false, as shown below.

**Theorem 3.** Let  $\{x_1^*, x_2^*, \dots, x_p^*\}$  be a  $p$ -periodic orbit of the function  $f$ , then the CML given by

$$X_i(n+1) = (1-\alpha)f(X_i(n-1)) + \frac{\alpha}{p} \sum_{i=1}^p f(X_i(n-1)) \quad i = 1, \dots, p$$

does not have a  $p$ -periodic orbit of the form

$$(X_1(n), X_2(n), \dots, X_p(n)) = (x_j^*, f(x_j^*), \dots, f^{p-1}(x_j^*))$$

$x_j^*$  being any of the  $p$  points of the  $p$ -periodic orbit.

### Proof

The following initial conditions are taken

$$(X_1(n), X_2(n), \dots, X_p(n)) = (x_1^*, x_2^*, \dots, x_p^*)$$



After the first iteration, it will become

$$\left\{ \begin{array}{l} X_1(n+1) = x_2^* = (1-\alpha)f(x_1^*) + \frac{\alpha}{p}(f(x_1^*) + \dots + f(x_p^*)) \\ X_2(n+1) = x_3^* = (1-\alpha)f(x_2^*) + \frac{\alpha}{p}(f(x_1^*) + \dots + f(x_p^*)) \\ \dots \\ X_p(n+1) = x_1^* = (1-\alpha)f(x_p^*) + \frac{\alpha}{p}(f(x_1^*) + \dots + f(x_p^*)) \end{array} \right.$$

therefore:

$$\left\{ \begin{array}{l} x_2^* = (1-\alpha)x_2^* + \frac{\alpha}{p}(x_1^* + \dots + x_p^*) \\ x_3^* = (1-\alpha)x_3^* + \frac{\alpha}{p}(x_1^* + \dots + x_p^*) \\ \dots \\ x_1^* = (1-\alpha)x_1^* + \frac{\alpha}{p}(x_1^* + \dots + x_p^*) \end{array} \right.$$

operating it results in:

$$\left\{ \begin{array}{l} \alpha x_2^* = \frac{\alpha}{p}(x_1^* + \dots + x_p^*) \\ \alpha x_3^* = \frac{\alpha}{p}(x_1^* + \dots + x_p^*) \\ \dots \\ \alpha x_1^* = \frac{\alpha}{p}(x_1^* + \dots + x_p^*) \end{array} \right.$$

from which it is deduced that

$$x_1^* = x_2^* = \dots = x_p^*$$

in contradiction with  $x_1^* \neq x_2^* \neq \dots \neq x_p^*$ .

This negative result, about  $p$ -periodic waves, brings us to question the conditions under which they are produced. This study is conducted in the following section.

## 2 Analytical study of patterns in weakly coupled CML

### 2.1 Travelling waves

It has been proven in Theorem 3 that the  $p$ -periodic orbit of the function  $f(x; r)$  is not inherited by the system, but it is easily observed that if  $\alpha = 0$  then a wave of this period exists in the CML. Given that for  $\alpha = 0$  the wave exists, one would wonder if for a small coupling  $\alpha \ll 1$ , the CML admits a perturbative solution. For this study, we will substitute  $\alpha$  with  $\varepsilon\alpha$ , having  $\varepsilon \ll 1$  and assuming the new  $\alpha$  is  $O(1)$ , in the CML given by (1).

**Theorem 4.** Let  $\{x_1^*, x_2^*, \dots, x_p^*\}$  be a  $p$ -periodic orbit of a  $C^2$  function  $f$ , such that  $f^{p'}(x_i^*) \neq 1$ ,  $i = 1, \dots, p$ , then the CML given by

$$X_i(n+1) = (1 - \varepsilon\alpha)f(X_i(n)) + \frac{\alpha\varepsilon}{p} \sum_{j=1}^p f(X_j(n)) \quad (5)$$

$$i = 1, \dots, p \quad \varepsilon \ll 1$$

shows a  $p$ -periodic solution given by

$$X_i(n+j) = x_{i+j}^* + \varepsilon A_{i+j}$$

$$i = 1, \dots, p$$

$$j = 0, \dots, p-1$$

where

$$A_k = \frac{\alpha}{(1 - (f^p(x_1))')} \sum_{j=1}^p [f^{p-j+k-1}(x_{j+1}^*)]' \left( (x_{j+1}^*) + \frac{1}{p} \sum_{l=1}^p x_l^* \right) \quad k = 1, \dots, p$$

with periodic conditions

$$\begin{aligned} A_{i+p} &= A_i \\ x_{i+p}^* &= x_i^* \end{aligned} \quad i = 1, \dots, p$$

### Proof

The periodic orbit given by

$$\begin{aligned} X_i(n+j) &= x_{i+j}^* + \varepsilon A_{i+j} \\ i &= 1, \dots, p \\ j &= 0, \dots, p-1 \end{aligned}$$

will exist when the following system

$$\left\{ \begin{aligned} X_i(n) &= x_i^* + \varepsilon A_i \\ X_i(n+1) &= x_{i+1}^* + \varepsilon A_{i+1} \\ &\vdots \\ X_i(n+p-1) &= x_{i+p-1}^* + \varepsilon A_{i+p-1} = x_{i-1}^* + \varepsilon A_{i-1} \\ X_i(n+p) &= x_i^* + \varepsilon A_i \end{aligned} \right. \quad i = 1, \dots, p \quad (6)$$

is compatible and determined.

As

$$X_i(n+1) = (1 - \varepsilon\alpha)f(X_i(n)) + \frac{\varepsilon\alpha}{p} \sum_{j=1}^p f(X_j(n))$$

from (6), it results in

$$x_{i+1}^* + \varepsilon A_{i+1} = (1 - \varepsilon\alpha)f(x_i^* + \varepsilon A_i) + \frac{\varepsilon\alpha}{p} \sum_{j=1}^p f(x_j^* + \varepsilon A_j) \quad (7)$$

Performing the expansion

$$f(x_i^* + \varepsilon A_i) = f(x_i^*) + \varepsilon A_i f'(x_i^*) + O(\varepsilon^2)$$

and replacing in (7), the system results in

$$x_{i+1}^* + \varepsilon A_{i+1} = x_{i+1}^* + \varepsilon A_i f'(x_i^*) - \varepsilon\alpha x_{i+1}^* + \frac{\varepsilon\alpha}{p} \sum_{j=1}^p (x_{j+1}^*) + O(\varepsilon^2)$$

$$i = 1, \dots, p$$

Solving the system to order  $\varepsilon$  it is obtained:

$$-A_i f'(x_i^*) + A_{i+1} = \alpha x_{i+1}^* + \frac{\alpha}{p} \sum_{j=1}^p x_{j+1}^* \quad i = 1, \dots, p$$

whose matricial expression is:

$$\begin{pmatrix} -f'(x_1^*) & 1 & 0 & 0 & \cdots & 0 \\ 0 & -f'(x_2^*) & 1 & 0 & \cdots & 0 \\ 0 & 0 & -f'(x_3^*) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & -f'(x_p^*) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_p \end{pmatrix} = \alpha \begin{pmatrix} -x_2^* + \frac{1}{p} \sum_{j=1}^p x_j^* \\ -x_3^* + \frac{1}{p} \sum_{j=1}^p x_j^* \\ -x_4^* + \frac{1}{p} \sum_{j=1}^p x_j^* \\ \vdots \\ -x_1^* + \frac{1}{p} \sum_{j=1}^p x_j^* \end{pmatrix} \quad (8)$$

This is a system of  $p$  equations and  $p$  unknowns whose coefficient matrix has determinant

$$(-1)^p \prod_{i=1}^p f'(x_i^*) + (-1)^{p+1}$$

Given that  $\prod_{i=1}^p f'(x_i^*) = f^{p'}(x_i^*) \neq 1$  (by hypothesis) the system is compatible and determined for every  $\alpha \neq 0$ . Moreover, the solution of the system is different from the trivial one, since the independent term column is not null  $x_1^* \neq x_2^* \neq \dots \neq x_p^*$ , that is

$$\begin{pmatrix} -x_2^* + \frac{1}{p} \sum_{j=1}^p x_j^* \\ -x_3^* + \frac{1}{p} \sum_{j=1}^p x_j^* \\ -x_4^* + \frac{1}{p} \sum_{j=1}^p x_j^* \\ \vdots \\ -x_1^* + \frac{1}{p} \sum_{j=1}^p x_j^* \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

It does not matter which oscillator is considered for study of the evolution of the system, as the algebraic system obtained is always the same.

The solution of the system in (8) can be obtained directly by inversion and results in:

$$\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_p \end{pmatrix} = \alpha \begin{pmatrix} -f'(x_1^*) & 1 & 0 & \cdots & 0 \\ 0 & -f'(x_2^*) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -f'(x_p^*) \end{pmatrix}^{-1} \begin{pmatrix} -x_2^* + \frac{1}{p} \sum_{j=1}^p x_j^* \\ -x_3^* + \frac{1}{p} \sum_{j=1}^p x_j^* \\ \vdots \\ -x_1^* + \frac{1}{p} \sum_{j=1}^p x_j^* \end{pmatrix}$$

The inversion of the matrix (which is not a circulant one) results in the following

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_p \end{pmatrix} = \alpha \frac{1}{(-1)^{p+1}(1 - (f^p(x_1^*))')} MN \quad (9)$$

where the matrix  $M$  is given by

$$M = \begin{pmatrix} f'(x_2^*) \cdots f'(x_p^*) & f'(x_3^*) \cdots f'(x_p^*) & f'(x_4^*) \cdots f'(x_p^*) & \cdots & 1 \\ 1 & f'(x_3^*) \cdots f'(x_p^*) f'(x_1^*) & f'(x_4^*) \cdots f'(x_p^*) f'(x_1^*) & \cdots & f'(x_1^*) \\ f'(x_2) & 1 & f'(x_4^*) \cdots f'(x_p^*) f'(x_1^*) f'(x_2^*) & \cdots & f'(x_1^*) f'(x_2^*) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f'(x_2^*) \cdots f'(x_{p-1}^*) & f'(x_3^*) \cdots f'(x_{p-1}^*) & f'(x_4^*) \cdots f'(x_{p-1}^*) & \cdots & f'(x_1^*) f'(x_2^*) \cdots f'(x_{p-1}^*) \end{pmatrix} \quad (10)$$

and  $N$  by

$$N = \begin{pmatrix} -x_2^* + \frac{1}{p} \sum_{j=1}^p x_j^* \\ -x_3^* + \frac{1}{p} \sum_{j=1}^p x_j^* \\ -x_4^* + \frac{1}{p} \sum_{j=1}^p x_j^* \\ \vdots \\ -x_1^* + \frac{1}{p} \sum_{j=1}^p x_j^* \end{pmatrix}$$

After operating in (9) it results in:

$$A_k = \frac{\alpha}{(-1)^{p+1}(1-(f^p(x_1))')} \sum_{j=1}^p [f^{p-j+k-1}(x_{j+1}^*)]' \left( (x_{1+j}^*) + \frac{1}{p} \sum_{l=1}^p x_l^* \right) \quad (11)$$

$$k = 1, \dots, p$$

Every  $A_k \neq 0$  because the solution is known to be different from the trivial one.

The solution obtained is valid at order  $O(\varepsilon^2)$  while  $\varepsilon \ll \frac{1}{1-f^{p'}(x_1^*)}$ .

## 2.2 Period doubling cascade for travelling waves in a CML

Period-doubling transitions to chaos have already been observed a long time ago, in CML with nearest neighbour coupling, using the Mandelbrot map [14]. The existence of this phenomenon is not relegated only to the quadratic functions, and its existence can be proved for any function (as we will demon-

strate) undergoing a period-doubling cascade; therefore, this phenomenon must be very frequent.

**Theorem 5.** Let  $f : I \rightarrow I$  be a  $C^2$  function depending on some parameter, in function of which the  $2^p$ -periodic orbit of the map  $x_{n+1} = f(x_n)$  undergoes a period-doubling cascade. Let  $\{x_{i,2^{p+q}}^*\}_{i=1}^{2^{p+q}}$  be the  $2^{p+q}$ -period orbit of the cascade,  $q \in \mathbb{N}$ , where it is noted that  $f^k(x_{i,2^{p+q}}^*) = x_{i+k,2^{p+q}}^*$ .

The CML given by

$$\begin{aligned} X_i(n+1) &= (1 - \varepsilon\alpha)f(X_i(n)) + \frac{\alpha\varepsilon}{2^p} \sum_{j=1}^{2^p} f(X_j(n)) \\ i &= 1, \dots, 2^p \quad \varepsilon \ll 1 \end{aligned} \tag{12}$$

has a  $2^{p+q}$ -periodic solution given by

$$\begin{aligned} X_i(n+j) &= x_{2^q(i-1)+1+j,2^{p+q}}^* + \varepsilon A_{2^q(i-1)+1+j} \\ i &= 1, \dots, 2^p \quad j = 0, \dots, 2^{p+q} \end{aligned}$$

where

$$\begin{aligned} A_k &= \frac{\alpha}{(-1 + (f^{2^{p+q}}(x_1^*))')} \sum_{j=1}^{2^{p+q}} \left[ f^{2^{p+q}-j+k-1}(x_{j+1,2^{p+q}}^*) \right]' \left( (x_{1+j,2^{p+q}}^*) + \frac{1}{p} S_j \right) \\ k &= 1, \dots, 2^{p+q} \end{aligned}$$



with

$$S_j = \begin{cases} \sum_{i=1}^{2^p} x_{2^q(i-1)+2, 2^{p+q}}^* & \text{if } j = [2^q] \\ \sum_{i=1}^{2^p} x_{2^q(i-1)+3, 2^{p+q}}^* & \text{if } j = [2^q] + 1 \\ \sum_{i=1}^{2^p} x_{2^q(i-1)+4, 2^{p+q}}^* & \text{if } j = [2^q] + 2 \\ \sum_{i=1}^{2^p} x_{2^q(i-1)+5, 2^{p+q}}^* & \text{if } j = [2^q] + 3 \\ \vdots & \vdots \\ \sum_{i=1}^{2^p} x_{2^q(i-1)+2^q+1, 2^{p+q}}^* & \text{if } j = [2^q] + 2^q - 1 \end{cases}$$

where  $[2^q]$  represents a multiple of  $2^q$ . This  $2^{p+q}$ -periodic solution fulfills the periodicity condition

$$\begin{aligned} A_{i+2^{p+q}} &= A_i \\ x_{i+2^{p+q}, 2^{p+q}}^* &= x_{i, 2^{p+q}}^* \end{aligned}$$

## Proof

**Note:** Notice that although there are just  $2^p$  oscillators the periodic wave will have period  $2^{p+q}$ , therefore, the system that is dealt with in the proof will have  $2^{p+q}$  equations.

Let  $2^p$  oscillators be with the initial conditions given by

$$X_i(n) = x_{2^q(i-1)+1, 2^{p+q}}^* + \varepsilon A_{2^q(i-1)+1} \quad i = 1, \dots, 2^p \quad (13)$$

Initial conditions are fixed points of  $f^{2^{p+q}}$  (taken one every  $2^q$ ) plus a

perturbation that must be calculated. A  $2^{p+q}$ -periodic orbit will exist whenever the system

$$\left\{ \begin{array}{lcl} X_i(n) & = & x_{2^q(i-1)+1, 2^{p+q}}^* + \varepsilon A_{2^q(i-1)+1} \\ X_i(n+1) & = & x_{2^q(i-1)+2, 2^{p+q}}^* + \varepsilon A_{2^q(i-1)+2} \\ & \vdots & \\ X_i(n+2^{p+q}) & = & x_{2^q(i-1)+1+2^{p+q}, 2^{p+q}}^* + \varepsilon A_{2^q(i-1)+2^{p+q}} \\ & = & x_{2^q(i-1)+1, 2^{p+q}}^* + \varepsilon A_{2^q(i-1)+1} = X_i(n) \end{array} \right. \quad i = 1, \dots, 2^p \quad (14)$$

is compatible and determined.

From equation (12) and substituting the second equality in (14), we have

$$\begin{aligned} x_{2^q(i-1)+2, 2^{p+q}}^* + \varepsilon A_{2^q(i-1)+2} &= (1 - \varepsilon \alpha) f(x_{2^q(i-1)+1, 2^{p+q}}^* + \varepsilon A_{2^q(i-1)+1}) + \\ &+ \frac{\alpha \varepsilon}{2^p} \sum_{j=1}^{2^p} f(x_{2^q(j-1)+1, 2^{p+q}}^* + \varepsilon A_{2^q(j-1)+1}) \end{aligned} \quad (15)$$

Performing a Taylor expansion of  $f$  to order  $O(\varepsilon^2)$  and substituting in (15) the following is obtained

$$\begin{aligned} x_{2^q(i-1)+2, 2^{p+q}}^* + \varepsilon A_{2^q(i-1)+2} &= x_{2^q(i-1)+2, 2^{p+q}}^* + \\ &+ \varepsilon A_{2^q(i-1)+1} f'(x_{2^q(i-1)+1, 2^{p+q}}^*) - \varepsilon \alpha x_{2^q(i-1)+2, 2^{p+q}}^* + \frac{\varepsilon \alpha}{2^p} \sum_{i=1}^{2^p} x_{2^q(i-1)+2, 2^{p+q}}^* + O(\varepsilon^2) \end{aligned}$$

Doing exactly the same with the next equality in (14) the following is obtained:

$$\begin{aligned} x_{2^q(i-1)+3, 2^{p+q}}^* + \varepsilon A_{2^q(i-1)+3} &= x_{2^q(i-1)+3, 2^{p+q}}^* + \\ &+ \varepsilon A_{2^q(i-1)+2} f'(x_{2^q(i-1)+2, 2^{p+q}}^*) - \varepsilon \alpha x_{2^q(i-1)+3, 2^{p+q}}^* + \frac{\varepsilon \alpha}{2^p} \sum_{i=1}^{2^p} x_{2^q(i-1)+3, 2^{p+q}}^* + O(\varepsilon^2) \end{aligned}$$

and with the last equality in (14), we get the following equation:

$$x_{2^q(i-1)+1, 2^{p+q}}^* + \varepsilon A_{2^q(i-1)+1} = x_{2^q(i-1)+1, 2^{p+q}}^* + \\ + \varepsilon A_{2^q(i-1)} f'(x_{2^q(i-1), 2^{p+q}}^*) - \varepsilon \alpha x_{2^q(i-1)+1, 2^{p+q}}^* + \frac{\varepsilon \alpha}{2^p} \sum_{i=1}^{2^p} x_{2^q(i-1)+1, 2^{p+q}}^* + O(\varepsilon^2)$$

The former  $2^{p+q}$ -equations, for the oscillator i, represent a linear system, whose matricial expression is:

$$\begin{pmatrix} -f'(x_{2^q(i-1)+1, 2^{p+q}}^*) & 1 & 0 & \cdots & 0 \\ 0 & -f'(x_{2^q(i-1)+2, 2^{p+q}}^*) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -f'(x_{2^q(i-1)+2^{p+1}, 2^{p+q}}^*) \end{pmatrix} \cdot \begin{pmatrix} A_{2^q(i-1)+1} \\ A_{2^q(i-1)+2} \\ \vdots \\ A_{2^q(i-1)+2^{p+q}} \end{pmatrix} = \\ = \alpha \begin{pmatrix} -x_{2^q(i-1)+2, 2^{p+q}}^* + \frac{1}{2^p} \sum_{i=1}^{2^{p+1}} x_{2^q(i-1)+2, 2^{p+q}}^* \\ -x_{2^q(i-1)+3, 2^{p+q}}^* + \frac{1}{2^p} \sum_{i=1}^{2^{p+1}} x_{2^q(i-1)+3, 2^{p+q}}^* \\ \vdots \\ -x_{2^q(i-1)+1, 2^{p+q}}^* + \frac{1}{2^p} \sum_{i=1}^{2^{p+1}} x_{2^q(i-1)+1, 2^{p+q}}^* \end{pmatrix} \quad (16)$$

The previous matricial expression represents a system of  $2^{p+q}$  equations with  $2^{p+q}$  unknowns, and being the determinant of coefficient matrix

$$(-1)^{2^{p+q}} \prod_{i=1}^{2^{p+q}} f'(x_i^*) - (-1)^{2^{p+q}} = \left[ f^{2^{p+q}}(x_i^*) \right]' - 1 \neq 0$$

(period doubling bifurcations take place when  $\left[ f^{2^{p+q}}(x_i^*) \right]' = -1$ ).

Thus the system is compatible and determined for every  $\alpha$  and, as in the previous cases, its solution is different from the trivial one for  $\alpha \neq 0$ .

The solution is obtained directly from (16) by inversion:

$$\begin{pmatrix} A_{2^q(i-1)+1} \\ A_{2^q(i-1)+2} \\ \vdots \\ A_{2^q(i-1)+2^{p+q}} \end{pmatrix} = \frac{\alpha}{(-1+[f^{2^{p+q}}(x_1^*)]')} MN \quad (17)$$

where

$$M = \begin{pmatrix} -f'(x_{2^q(i-1)+1,2^{p+q}}^*) & 1 & 0 & \cdots & 0 \\ 0 & -f'(x_{2^q(i-1)+2,2^{p+q}}^*) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -f'(x_{2^q(i-1)+2^{p+q},2^{p+q}}^*) \end{pmatrix}^{-1}$$

has already been calculated (see 10) and

$$N = \begin{pmatrix} -x_{2^q(i-1)+2,2^{p+q}}^* + \frac{1}{2^p} \sum_{i=1}^{2^{p+1}} x_{2^q(i-1)+2,2^{p+q}}^* \\ -x_{2^q(i-1)+3,2^{p+q}}^* + \frac{1}{2^p} \sum_{i=1}^{2^{p+1}} x_{2^q(i-1)+3,2^{p+q}}^* \\ \vdots \\ -x_{2^q(i-1)+1,2^{p+q}}^* + \frac{1}{2^p} \sum_{i=1}^{2^{p+1}} x_{2^q(i-1)+1,2^{p+1}}^* \end{pmatrix}$$

After operating in (17) it results in:

$$A_k = \frac{\alpha}{(-1)^{2^{p+q}}(1 - [f^{2^{p+q}}(x_1^*)]')} \sum_{j=1}^{2^{p+q}} \left[ f^{2^{p+q}-j+k-1}(x_{j+1,2^{p+q}}^*) \right]' \left( (x_{1+j,2^{p+q}}^*) + \frac{1}{p} S_j \right)$$

$$k = 1, \dots, 2^{p+q}$$

with

$$S_j = \begin{cases} \sum_{i=1}^{2^p} x_{2^q(i-1)+2, 2^{p+q}}^* & \text{if } j = [2^q] \\ \sum_{i=1}^{2^p} x_{2^q(i-1)+3, 2^{p+q}}^* & \text{if } j = [2^q] + 1 \\ \sum_{i=1}^{2^p} x_{2^q(i-1)+4, 2^{p+q}}^* & \text{if } j = [2^q] + 2 \\ \sum_{i=1}^{2^p} x_{2^q(i-1)+5, 2^{p+q}}^* & \text{if } j = [2^q] + 3 \\ \vdots & \vdots \\ \sum_{i=1}^{2^p} x_{2^q(i-1)+2^q+1, 2^{p+q}}^* & \text{if } j = [2^q] + 2^q - 1 \end{cases}$$

where  $[2^q]$  represents a multiple of  $2^q$ , with the periodicity conditions

$$A_{i+2^{p+q}} = A_i$$

$$x_{i+2^{p+q}, 2^{p+q}}^* = x_{i, 2^{p+q}}^*$$

due to the cyclic character of the  $2^{p+q}$ -periodic orbit.

The solution obtained is valid at order  $O(\varepsilon^2)$  while  $\varepsilon \ll \frac{1}{1-[f^{2^{p+q}}(x_1^*)]}'}$ .

The saddle-node orbit  $([f^{2^{p+q}}(x_1^*)]' \neq 1)$  has been avoided by the condition  $q \in \mathbb{N}$  ( $0 \notin \mathbb{N}$ ), and therefore there has been, at least, one period-doubling bifurcation.

**Remarks:**

1. Notice that this theorem indicates that a CML, with  $2^p$  oscillators, has originally a  $2^p$ -period travelling wave. As the  $2^p$ -periodic orbit of  $f$  duplicates ( $q$  increases) to a  $2^{p+q}$ -periodic orbit so does the travelling wave of the CML.
2. The theorem 5 does not impose any restriction to the  $2^p$ -periodic orbit of  $f$ . In the case of  $f$  presenting  $2^p$ -periodic windows, this  $2^p$ -periodic orbit could belong to a period-doubling cascade in the canonical window, or originate from a  $2^p$ -periodic saddle-node orbit. In the former case,  $f$  undergoes a period-doubling cascade, in the latter it is  $f^{2^p}$  who goes through a period-doubling cascade (this period-doubling cascade would be located inside a  $2^p$ -periodic window). The conclusion is straightforward: the CML will not have just one  $2^p$ -periodic wave, undergoing a period-doubling cascade; there will be as many as  $2^{p_1}$ -periodic windows of  $f$ , with  $p_1 \leq p$  ( $p_1 = 0$  would be the canonical window).
3. Similar arguments can be done for synchronized state cascades, subject to the condition that the duplicating orbit does not have prime period: it can be a  $p \cdot q$ -periodic orbit in the canonical window, or a  $q$ -periodic orbit in the  $p$ -periodic window, or a  $p$ -periodic orbit in a  $q$ -periodic window, that afterwards will undergo period bifurcation cascade.
4. There is a fact that could be not observed at a first reading of theorem 5: the points used to construct the perturbative solution both can be

stable and unstable.

5. Since it has been deduced that the CML undergoes a period doubling cascade and that this cascade has its origin in the period doubling cascade of the  $f$ , it is concluded that the CML inherits the dynamics of  $f$ .

### 3 Discussion and conclusions

Several theorems have been proved that show the sufficiency conditions of existence of synchronized states (periodic and chaotic) and travelling waves in CML. Also it has been analytically determined the value that describes the state of each oscillator at any moment. The results of the theorems are as general as possible. This is due to two facts. Firstly, the CML, with which we have worked, has a number of arbitrary oscillators. Second, the function  $f$ , that rules the dynamics of every oscillator, is also arbitrary with the condition that it undergoes a period-doubling cascade.

The results have the following consequences and link with other research:

- i) The emergence of the global properties from the local ones has been proved. The global dynamics inherits the dynamics of every oscillator: fixed points of the system come essentially from the fixed points of the map (that governs the dynamics of every oscillator) compounded with itself  $m \cdot 2^k$  times (being  $m$  number of oscillators of the CML). In particular, this result has been observed recently in numerical computations [16].

Our results are an explicit analytical expression of the results of Lemaitré y Chaté [15], who proved, in CML, the translation of the local properties to a spatiotemporal level.

- ii) The dynamics of a CML has been studied, where the individual dynamics of every oscillator is ruled by an arbitrary function  $f$ ; being few the analytical results on the matter, one normally only works with quadratic functions or piece-wise linear functions [10, 12, 17]. The one presented here is an interesting generalization that permits the calculation of properties associated with the states and their evolution.
- iii) Two limitations that are present when numerical techniques are applied have been overcome:
  - (a) Spurious results, due to finite precision in computer simulations [18, 19, 20], have been avoided.
  - (b) Limits tending to infinity can be used for analytical solutions, both with the number of oscillators in CML and the number of bifurcations in the period doubling cascade.

On the one hand, the numerical simulation with a large number of oscillators becomes unaffordable due to the computation time that would be necessary as the number of oscillators grows. On the other hand, as it has been indicated in the introduction, the study of the onset of the turbulence in a fluid to be properly understood would need many oscillators, the more the better.

As a direct consequence, of taking the limit in the period doubling



cascade, it is deduced the existence of waves of arbitrary period, tending to infinity as the parameter bifurcation gets closer to the Myrberg-Feigenbaum point. This is a response to the established question of Gade and Amritkar in their work [21] where they found the wavelength-doubling bifurcation. Another question raised by Gade and Amritkar in that same paper was: is there more than one value such that, if the parameter value tends to it, then the period of the travelling wave tends to infinity?. The response again in the affirmative; in fact, there are infinite values that are the correspondent Myrberg-Feigenbaum points of the windows inside the canonical window. The position of these values is determined by the Saddle-Node Bifurcation Cascades [22] and the relation between period doubling cascade and Saddle-Node Bifurcation Cascades is also known [23].

## References

- [1] Kaneko K. Theory and applications of coupled map lattices. New York: Wiley; 1993.
- [2] Kaneko K. Chaotic but Regular Posi-nega Switch among Coded Attractors by Cluster Size Variation. Phys. Rev. Lett. 1989; 63: 219-224.
- [3] Kaneko K. Clustering, Coding, Switching, Hierarchical Ordering, and Control in Network of Chaotic Elements. Physica D 1990; 41: 131-172.

- [4] Kaneko K. Globally Coupled Chaos Violates Law of Large Numbers. Phys. Rev. Lett. 1990; 65: 1391-1394.
- [5] Kaneko K. Partition Complexity in Network of Chaotic Elements. J. Phys. A 1991; 24: 2107-2119.
- [6] Kaneko K. Globally Coupled Circle Maps. Physica D 1991; 54: 5-19
- [7] [Special issue. Physica D 1997:103]
- [8] [Special issue: Chaos 1992;2(3)]
- [9] Chaté H, Manneville P. Spatio-temporal intermittency in coupled map lattices. Physica D 1988; 32:409-422.
- [10] Atmanspacher H, Scheingraber H. Inherent global stabilization of unstable local behavior in coupled map lattices. Int J Bifurcat Chaos 2005; 5(15):1665–1676.
- [11] Mehta M, Sinha S. Asynchronous updating of coupled maps leads to synchronization. Chaos 2000; 10:350-358.
- [12] Anteneodo C, de S. Pinto SE, Batista AM, Viana RL. Analytical results for coupled-map lattices with long-range interactions. Phys. Rev. E 2003; 68:045202(R) [Erratum Phys. Rev. E 2004; 69:029904]
- [13] Bohr T, Christensen OB. Size dependence, coherence and scaling in turbulent coupled map lattices. Phys. Rev. Lett. 1989; 63:2161-2164

- [14] Willeboordse FH. Selection of Windows, Attractors and Self-similar Patterns in a Coupled Map Lattice. *Chaos, Solitons and Fractals* 1992; 2:609-634.
- [15] Lemaître A, Chaté H. Nonperturbative Renormalization Group for Chaotic Coupled Map Lattices. *Phys. Rev. Lett.* 1998; 80(25):5528-5531.
- [16] Palaniyandi P, Muruganandam P, Laksmanan M. Coexistence of synchronized and desynchronized patterns in coupled chaotic dynamical systems. *Chaos, Solitons and Fractals* 2006;doi:10. 1016/j.chaos.2006.08.004
- [17] Li P, Li Z, Halang WA, Chen G. Li-Yorke chaos in a spatiotemporal chaotic system. *Chaos, Solitons and Fractals* 2007; 33: 335-341.
- [18] Grebogi C, Hammel SM, Yorke JA, Sauer T. Shadowing of physical trajectories in chaotic dynamics: containment and refinement. *Phys. Rev. Lett.* 1990; 65:1527-1530.
- [19] Zhou C, Lai C-H. Analysis of spurious synchronization with positive conditional Lyapunov exponents in computer simulations. *Physica D* 2000; 135:1-23.
- [20] Zhou C, Kurths J. Noise-induced phase synchronization and synchronization transitions in chaotic oscillators. *Phys. Rev. Lett.* 2002; 88:230602.
- [21] Gade PM, Amritkar RE. Wavelength-doubling bifurcations in one-dimensional coupled logistic map. *Phys. Rev. E* 1994; 49(4):2617-2622.

- [22] San Martín J. Intermittency cascades. *Chaos Solitons and Fractals* 2007; 32:816-831.
- [23] San Martín J, Rodriguez-Perez D. Conjugation of cascades. *Chaos Solitons and Fractals*. (In Press doi:10.1016/J.Chaos.2007.01.073)